2019

ε-SUPERPOSITION AND TRUNCATION DIMENSIONS IN AVERAGE AND PROBABILISTIC SETTINGS FOR ∞-VARIATE LINEAR PROBLEMS

Jonathan M. Dingess
University of Kentucky, j_and_m33@yahoo.com
Digital Object Identifier: https://doi.org/10.13023/etd.2019.218

STUDENT AGREEMENT:

I represent that my thesis or dissertation and abstract are my original work. Proper attribution has been given to all outside sources. I understand that I am solely responsible for obtaining any needed copyright permissions. I have obtained needed written permission statement(s) from the owner(s) of each third-party copyrighted matter to be included in my work, allowing electronic distribution (if such use is not permitted by the fair use doctrine) which will be submitted to UKnowledge as Additional File.

I hereby grant to The University of Kentucky and its agents the irrevocable, non-exclusive, and royalty-free license to archive and make accessible my work in whole or in part in all forms of media, now or hereafter known. I agree that the document mentioned above may be made available immediately for worldwide access unless an embargo applies.

I retain all other ownership rights to the copyright of my work. I also retain the right to use in future works (such as articles or books) all or part of my work. I understand that I am free to register the copyright to my work.

REVIEW, APPROVAL AND ACCEPTANCE

The document mentioned above has been reviewed and accepted by the student's advisor, on behalf of the advisory committee, and by the Director of Graduate Studies (DGS), on behalf of the program; we verify that this is the final, approved version of the student's thesis including all changes required by the advisory committee. The undersigned agree to abide by the statements above.

Jonathan M. Dingess, Student

Dr. G. W. Wasilkowski, Major Professor

Dr. M. Truszczyński, Director of Graduate Studies

ε-SUPERPOSITION AND TRUNCATION DIMENSIONS IN AVERAGE AND PROBABILISTIC SETTINGS FOR ∞-VARIATE LINEAR PROBLEMS

THESIS

A thesis submitted in partial
fulfillment of the requirements for
the degree of Master of Science in
the College of Engineering at the
University of Kentucky

By
Jonathan Dingess
Lexington, Kentucky

Director: Dr. G. W. Wasilkowski, Professor of Computer Science
Lexington, Kentucky 2019

ABSTRACT OF THESIS

ε-SUPERPOSITION AND TRUNCATION DIMENSIONS IN AVERAGE AND PROBABILISTIC SETTINGS FOR ∞-VARIATE LINEAR PROBLEMS

This thesis is a representation of my contribution to the paper of the same name I co-author with Dr. Wasilkowski [3]. It deals with linear problems defined on $\boldsymbol{\gamma}$-weighted normed spaces of functions with infinitely many variables. In particular, I describe methods and discuss results for ε-truncation and ε-superposition methods. I show through these results that the ε-truncation and ε-superposition dimensions are small under modest error demand ε. These positive results are derived for product weights and the so-called *anchored* decomposition.

KEYWORDS: infinite-variate linear problems, epsilon-superposition, epsilon-truncation, product weights, multivariate decomposition methods, changing dimension algorithms

Author's signature: Jonathan Dingess

Date: May 15, 2019

ε-SUPERPOSITION AND TRUNCATION DIMENSIONS IN AVERAGE AND PROBABILISTIC SETTINGS FOR ∞-VARIATE LINEAR PROBLEMS

By
Jonathan Dingess

Director of Dissertation: G. W. Wasilkowski

Director of Graduate Studies: M. Truszczyński

Date: May 15, 2019

ACKNOWLEDGMENTS

This thesis represents my contribution to the ongoing work of Dr. G. W. Wasilkowski, and could not exist without his research and guidance. It was through him that I was introduced to this field and it was his papers and books I read for background knowledge when stuck at any point in my research. I thank him for this and for his patience with me when I was still learning. Next, I would like to thank Kathy Ice-Wedding for her guidance through the thesis process and for always being available for questions. Finally, I would also like to thank Dr. Jacobs and Dr. Zhang for serving on my defense committee and taking the time to review this document.

TABLE OF CONTENTS

Chapter 1 Introduction

This thesis is based on the paper [3] of the same name I co-authored with my advisor Dr. Wasilkowski. It focuses on my contributions to the project and I have had to borrow methods and sections from what Dr. Wasilkowski wrote in that paper, with his permission, in order to provide context for my contributions.

Our paper is concerned with the complexity of linear problems defined on $\boldsymbol{\gamma}$-weighted Hilbert spaces of functions with infinitely many variables. Such problems appear in a number of applications including stochastic differential equations and partial differential equations with random coefficients. See, e.g., [9] for a survey.

In our paper, we define and analyze new concepts of ε-superposition and ε-truncation dimensions in average case and probabilistic settings of *Information-Based Complexity* (IBC for short), see [15]. Roughly speaking, these concepts quantify how well the solutions for ∞-variate functions can be approximated by the solutions of special functions that depend on only few variables. Originally, these concepts were considered in, e.g., [1, 16], however the dimensions were defined for specific functions or for very special subclasses of functions.

This thesis defines and discusses efficient methods for calculating ε-superposition and ε-truncation dimensions in average and probabilistic settings. The results of these methods and an analysis of time complexity is included.

We consider reproducing kernel Hilbert spaces $\mathcal{F}_\infty$ of functions f with infinitely many variables that admit unique decompositions

$$f \;=\; \sum_{\substack{\mathfrak{u}\subset\mathbb{N}\\ |\mathfrak{u}|\le\infty}} f_{\mathfrak{u}},$$

where $f_{\mathfrak{u}}$ is a function that depends only on the variables in $\mathfrak{u}$. For example, if $\mathfrak{u}=\{1,2,5,29\}$, then $f_{\mathfrak{u}}$ is a function depending on only the first, second, fifth, and twenty-ninth variables from f. For $\mathfrak{u}=\emptyset$, $f_\emptyset$ is a constant function. Specifically, my work is concerned with *anchored* decompositions. A decomposition is anchored if $f_{\mathfrak{u}}(x_{\mathfrak{u}})=0$ if any $x_j=0$ with $j\in\mathfrak{u}$.

We endow $\mathcal{F}_\infty$ with a $\boldsymbol{\gamma}$-weighted norm

$$\|f\|_{\mathcal{F}_\infty} \;=\; \left[\sum_{\mathfrak{u}} (\gamma_{\mathfrak{u}}^{-1}\,\|f_{\mathfrak{u}}\|_{F_{\mathfrak{u}}})^2\right]^{1/2},$$

where $\gamma_{\mathfrak{u}}$ are non-negative numbers, called *weights*, and $F_{\mathfrak{u}}$ are Hilbert spaces of functions of $|\mathfrak{u}|$ variables. In our paper we consider other weights, but my work and thus this thesis is only concerned with weights in the *product form*

$$\gamma_{\mathfrak{u}} \;=\; \prod_{j\in\mathfrak{u}} \gamma_j$$

introduced in [14]. I assume without loss of generality that

$$\gamma_j \;\ge\; \gamma_{j+1} \;>\; 0 \quad \text{for all } j\in\mathbb{N}.$$

The problem we consider is the approximation of $\mathcal{S}_\infty(f)$, where $\mathcal{S}_\infty$ is a linear and continuous operator from $\mathcal{F}_\infty$ into a normed space $\mathcal{G}$,

$$\mathcal{S}_\infty : \mathcal{F}_\infty \to \mathcal{G}.$$

A common example for such an $\mathcal{S}_\infty$ is definite integration of $f \in \mathcal{F}_\infty$ with $\mathcal{G} = \mathbb{R}$.

Our paper discusses two approaches to deal with such problems with ∞-varied functions. The first approach is to reduce the infinite number of variables to a finite (and possibly small) number k of variables. This leads us to the concept of *ε-truncation dimension* discussed in Chapter 3. The second approach is to reduce the decomposition $f = \sum_{\mathfrak{u}} f_u$ to $f = \sum_{\mathfrak{u}\in\mathcal{V}} f_u$, where each $\mathfrak{u} \in \mathcal{V}$ has at most a small number k of variables, and often $\mathcal{V}$ has low cardinality as well. This number k is referred to as the *ε-superposition dimension*, and is discussed in Chapter 4.

Both these concepts are similar to those in Financial Mathematics and Statistics, (see, e.g. [1, 16],) but unlike those fields, ε-truncation and ε-superposition in IBC are defined in the worst case setting with respect to the whole space $\mathcal{F}_\infty$. This worst-case approach to ε-truncation was first considered in [7], see also [6, 8]. The methods described in Chapter 3 are based on [7]. The presentation of ε-superposition in our paper is based on [5].

For a given error demand $\varepsilon > 0$, the ε-truncation dimension (in the worst case setting) is given by

$$\dim^{\text{trnc}}(\varepsilon) = \min\left(k : \sup_{f\in\mathcal{F}_\infty} \frac{\|\mathcal{S}_\infty(f - f_k)\|_{\mathcal{G}}}{\|f\|_{\mathcal{F}_\infty}} \le \varepsilon\right),$$

where

$$f_k(x_1, \ldots, x_k) := f(x_1, \ldots, x_k, 0, 0, \ldots).$$

Hence the ε-truncation dimension depends on the error ε, the solution operator S_∞, and the whole space $\mathcal{F}_\infty$, but does not depend on specific functions. It is enough to work with functions f_k with only $k = \dim^{\text{trnc}}(\varepsilon)$ variables to approximate the solutions $\mathcal{S}_\infty(f)$ for functions with ∞-many variables.

Denoting $\|\mathcal{V}\|_\infty = \max_{\mathfrak{u}\in\mathcal{V}} |\mathfrak{u}|$, the ε-superposition dimension (in the worst case) is given by

$$\dim^{\text{sprp}}(\varepsilon) = \min\left(\|\mathcal{V}\|_\infty : \sup_{f\in\mathcal{F}_\infty} \frac{\|\mathcal{S}(f) - \sum_{\mathfrak{u}\in\mathcal{V}} \mathcal{S}(f_{\mathfrak{u}})\|_{\mathcal{F}}}{\|f\|_{\mathcal{F}_\infty}} \le \varepsilon\right).$$

This requires the decomposition discussed earlier

$$f = \sum_{\substack{\mathfrak{u}\subseteq\mathbb{N} \\ |\mathfrak{u}|\le\infty}} f_{\mathfrak{u}}.$$

The superposition and truncation methods in the worst case have previously been considered in [5, 6, 7, 8, 13]. In particular, it was shown in [13] that the superposition dimension is as small as

$$\dim^{\text{sprp}}(\varepsilon, \mathcal{S}) = O\left(\frac{\ln(1/\varepsilon)}{\ln(\ln(1/\varepsilon))}\right) \quad \text{as } \varepsilon \to 0$$

and that the corresponding set $\mathcal{V}$ is relatively small. In [5] is an efficient algorithm for constructing the smallest possible $\mathcal{V}$ for product weights. I extend this algorithm to the average and probabilistic cases in Chapter 4.

The results discussed so far deal with the worst case setting. The first paper dealing with the average ε-truncation dimension is [8]. However, it was done for very special classes of ∞-variate functions. In [3], we consider more general γ-weighted Hilbert spaces $\mathcal{F}_\infty$ that are endowed with zero-mean Gaussian measures μ_∞. Our paper defines the *average ε-truncation dimension* and *probabilistic (ε,δ)-truncation dimension* as follows:

$$\dim^{\text{avg-t}}(\varepsilon) \;=\; \inf\left(k \;:\; \int_{\mathcal{F}_\infty} \|\mathcal{S}_\infty(f-f_k)\|_{\mathcal{G}}^2 \,\mu_\infty(\mathrm{d}f) \;\le\; \varepsilon^2\right)$$

and

$$\dim^{\text{prb-t}}(\varepsilon,\delta) \;=\; \inf\left(k \;:\; \mu_\infty\left(\{f \;:\; \|\mathcal{S}_\infty(f-f_k)\|_{\mathcal{G}} \;\le\; \varepsilon\}\right) \;\ge 1-\delta\right).$$

Here f_k is as in the definition of the worst case truncation dimension. I define and implement an efficent algorithm to calculate this dimension in Chapter 3.

The *average $\varepsilon-$ and probabilistic $(\varepsilon,\delta)-$superposition* dimensions are defined in a similar way. Chapter 4 details my method for calculating them.

The spaces $\mathcal{F}_\infty$ depend on weights $\boldsymbol{\gamma} = (\gamma_{\mathrm{u}})_{\mathrm{u}}$, and the measures μ_∞ depend on weights $\boldsymbol{\alpha} = (\alpha_{\mathrm{u}})_{\mathrm{u}}$, see Chapter 2. For the continuity of the solution operator, γ_{u}^2 has to be summable. For μ_∞ to be a Gaussian measure, $(\gamma_{\mathrm{u}}\,\alpha_{\mathrm{u}})^2$ have to be summable. Hence, the results depend on the speed of decay of γ_{u} and $\gamma_{\mathrm{u}}\,\alpha_{\mathrm{u}}$. We have computed the values of $\dim^{\text{avg-t}}$, $\dim^{\text{prb-t}}$, $\dim^{\text{avg-s}}$, and $\dim^{\text{prb-s}}$ for specific cases, some of them reported here.

Consider $\gamma_{\mathrm{u}} = \prod_{j\in\mathrm{u}} j^{-2}$ and $\alpha_{\mathrm{u}} = \prod_{j\in\mathrm{u}} j^{-\ell}$ for $\ell = 0,1,2$. Then for the standard L_∞ approximation and $\varepsilon = 10^{-i}$ for $i = 2,\ldots,5$ we have the following

$\varepsilon =$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$\ell = 0$	5	11	24	51
$\ell = 1$	3	6	10	18
$\ell = 2$	2	4	6	10

values of $\dim^{\text{avg-t}}(\varepsilon)$, and

$\varepsilon =$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$\ell = 0$	8	17	38	81
$\ell = 1$	4	8	14	25
$\ell = 2$	3	5	8	13

values of $\dim^{\text{prb-t}}(\varepsilon,\delta)$ for $\delta = 0.001$

We have even smaller superposition dimensions. In the following tables, we list the computed results for the same parameters as for the truncation dimension. Since the size of the corresponding set $\mathcal{V}$ is very important in practice, the values of the

dimensions are followed by the cardinalities of the constructed sets $\mathcal{V}$. We have

$\varepsilon =$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$\ell = 0$	3 and 12	3 and 40	4 and 134	4 and 424
$\ell = 1$	2 and 6	3 and 13	3 and 31	4 and 74
$\ell = 2$	2 and 4	2 and 7	3 and 14	3 and 28

for the average case, and

$\varepsilon =$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$\ell = 0$	3 and 25	4 and 84	4 and 277	5 and 886
$\ell = 1$	2 and 9	3 and 22	3 and 53	4 and 126
$\ell = 2$	2 and 5	3 and 11	3 and 21	3 and 42

for the probabilistic case with $\delta = 0.001$.

Chapter 2 Basic Definitions

As this thesis is more concerned with the computation than the definition of ε-superposition and ε-truncation methods, I have included only what is necessary to understand these methods. More detailed explanations can be found in [3].

2.1 Notation

I now introduce the notation used in this thesis. Let $\mathbb{N}$ denote the set of positive integers and $\mathcal{U}$ denote the collection of all finite and increasing sequences $\mathfrak{u}$ of positive integers of the form

$$\mathfrak{u} = (u_1, \dots, u_k), \quad \text{where} \quad u_i \in \mathbb{N} \quad \text{and} \quad u_j < u_{j+1}.$$

This includes the empty sequence $\emptyset$. It is often convenient to think of $\mathfrak{u}$ as a set. Then one can also perform set operations on them and write $k = |\mathfrak{u}|$ for cardinality of $\mathfrak{u}$.

Let D be a non-empty interval in $\mathbb{R}$ (possibly infinite like $\mathbb{R}_+ = [0, \infty)$ or $\mathbb{R} = (-\infty, \infty)$) and $D^{\mathbb{N}}$ be the set of infinite sequences $\boldsymbol{x} = (x_j)_{j \in \mathbb{N}}$ (sometimes referred to as *points*) with $x_j \in D$. We also assume that $0 \in D$ so that $\mathbf{0} = (0, 0, \dots) \in D^{\mathbb{N}}$.

For a given sequence $\mathfrak{u} = (u_1, \dots, u_k) \in \mathcal{U}$ with $k = |\mathfrak{u}|$ and point $\boldsymbol{x} \in D^{\mathbb{N}}$, we define

$$\boldsymbol{x}_{\mathfrak{u}} = (x_{u_1}, \dots, x_{u_k}) \qquad \text{and} \qquad [\boldsymbol{x}; \mathfrak{u}] = \boldsymbol{y} \quad \text{with} \quad y_j = \begin{cases} x_j & \text{if } j \in \mathfrak{u}, \\ 0 & \text{if } j \notin \mathfrak{u}. \end{cases}$$

In particular, $[\boldsymbol{x}; \emptyset] = \mathbf{0} = (0, 0, \dots)$.

2.2 γ-weighted spaces $\mathcal{F}_\infty$ and their measures μ_∞

Recall a Hilbert Space $\mathcal{H}$ of functions $f : D \to \mathbb{R}$ is a *Reproducing Kernel Hilbert Space* if the functionals $\delta_x(f) = f(x)$ are continuous for every $x \in D$. Moreover, the Riesz representation theorem implies that there exists a function $K : D \times D \to \mathbb{R}$, called the *Reproducing Kernel* of $\mathcal{H}$, with the following properties:

$$K(\cdot, x) \in \mathcal{H} \quad \text{and} \quad f(x) = \langle f, K(\cdot, x) \rangle_{\mathcal{H}} \qquad \text{for every } x \in D \text{ and every f } \in H.$$

Let F_1 be a reproducing kernel Hilbert space of functions $f : D \to \mathbb{R}$ whose reproducing kernel is denoted by K_1. We assume that the zero function is the only constant function in F_1, i.e.,

$$1 \notin F_1. \tag{2.1}$$

An important class of such spaces is provided by *anchored* spaces for which

$$K_1(0, 0) = 0 \quad \text{which is equivalent to} \quad f(0) = 0 \text{ for all } f \in F_1. \tag{2.2}$$

We endow F_1 with a zero-mean Gaussian measure μ_1 whose covariance operator and covariance kernel are denoted by C_1 and $C_1^{\ker}$, respectively. Recall that they are given by

$$\langle C_1(g), h\rangle_{F_1} \;=\; \int_{F_1} \langle f, g\rangle\, \langle f, h\rangle\; \mu_1(\mathrm{d}f) \quad \text{and} \quad C_1{}^{\ker}(x, y) \;=\; \int_{F_1} f(x)\, f(y)\, \mu_1(\mathrm{d}f)$$

and C_1 is symmetric, positive definite, and has a finite trace,

$$\mathrm{trace}(C_1) \;=\; \int_{F_1} \|f\|_{F_1}^2\, \mu_1(\mathrm{d}f) \;<\; \infty.$$

For $\mathfrak{u} = \emptyset$, let $F_\emptyset$ be the space of constant functions with the natural inner-product and the normal $\mathcal{N}(0,1)$ distribution as denoted later by $\mu_\emptyset$.

For non-empty $\mathfrak{u} \in \mathcal{U}$, let $F_\mathfrak{u}$ be the Hilbert space of functions $f : D^{\mathbb{N}} \to \mathbb{R}$ that depend only on $\boldsymbol{x}_\mathfrak{u}$, the variables listed in $\mathfrak{u}$. Its reproducing kernel is given by

$$K_\mathfrak{u}(\boldsymbol{x}, \boldsymbol{y}) \;=\; \prod_{j\in\mathfrak{u}} K_1(x_j, y_j),$$

and we endow $F_\mathfrak{u}$ with $\mu_\mathfrak{u}$, which is a zero-mean Gaussian measure with the covariance operator and kernel given respectively by $|\mathfrak{u}|$-fold tensor products of C_1 and $C_1^{\ker}$. In particular,

$$C_\mathfrak{u}^{\ker}(\boldsymbol{x}, \boldsymbol{y}) \;=\; \prod_{j\in\mathfrak{u}} C_1{}^{\ker}(x_j, y_j).$$

The average and probabilistic settings for such spaces and measures have been studied in many papers; see, e.g., [15].

We use them to define Hilbert spaces $\mathcal{F}_\infty$ of functions with ∞-many variables and corresponding Gaussian measures μ_∞ on them, similar as in [17].

Let $\boldsymbol{\gamma} = (\gamma_\mathfrak{u})_{\mathfrak{u}\in\mathcal{U}}$ be a family of nonnegative numbers called *weights*.

The space $\mathcal{F}_\infty$ is the Hilbert space spanned by $\bigoplus_{\mathfrak{u}\in\mathcal{U}} F_\mathfrak{u}$ with the norm given by

$$\|f\|_{\mathcal{F}_\infty}^2 \;=\; \sum_{\mathfrak{u}\in\mathcal{U}} \gamma_\mathfrak{u}^{-2}\, \|f_\mathfrak{u}\|_{F_\mathfrak{u}}^2 \quad \text{for} \quad f \;=\; \sum_{\mathfrak{u}\in\mathcal{U}} f_\mathfrak{u} \quad \text{with} \quad f_\mathfrak{u} \;\in\; F_\mathfrak{u}.$$

Note that the decompositions $f = \sum_{\mathfrak{u}\in\mathcal{U}} f_\mathfrak{u}$ is unique since (2.1) implies that

$$F_\mathfrak{u} \cap F_\mathfrak{v} \;=\; \{0\} \quad \text{if} \quad \mathfrak{u} \neq \mathfrak{v}.$$

Remark 1 *The space $\mathcal{F}_\infty$ is a reproducing kernel Hilbert space if and only if*

$$\sum_{\mathfrak{u}\in\mathcal{U}} \gamma_\mathfrak{u}^2\, K_\mathfrak{u}(\boldsymbol{x}, \boldsymbol{x}) \;<\; \infty \quad \textit{for every } \boldsymbol{x} \in D^{\mathbb{N}} \tag{2.3}$$

and then

$$\mathcal{K}_\infty(\cdot, \cdot) \;=\; \sum_{\mathfrak{u}\in\mathcal{U}} \gamma_\mathfrak{u}^2\, K_\mathfrak{u}(\cdot, \cdot)$$

is the reproducing kernel of $\mathcal{F}_\infty$. *If* (2.3) *does not hold, then for some* $\boldsymbol{x} \in D^{\mathbb{N}}$ *and* $f \in \mathcal{F}_\infty$, *the series* $\sum_{\mathrm{u}\in\mathcal{U}} f_{\mathrm{u}}(\boldsymbol{x})$ *does not converge. In such a case, we treat* $\mathcal{F}_\infty$ *as a space of sequences* $f = (f_{\mathrm{u}})_{\mathrm{u}\in\mathcal{U}}$. *However, even then*

$$f([\boldsymbol{x};\mathfrak{w}]) \;=\; \sum_{\mathrm{u}\in\mathcal{U}} f_{\mathrm{u}}([\boldsymbol{x};\mathfrak{w}]) \;=\; \sum_{\mathrm{u}\subseteq\mathfrak{w}} f_{\mathrm{u}}(\boldsymbol{x})$$

is finite for every $\boldsymbol{x} \in D^{\mathbb{N}}$, $\mathfrak{w} \in \mathcal{U}$, *and* $f \in \mathcal{F}_\infty$, *if the spaces* F_{u} *are anchored at zero.*

For a given family $\boldsymbol{\alpha} = (\alpha_{\mathrm{u}})_{\mathrm{u}\in\mathcal{U}}$ of positive numbers, we endow the space $\mathcal{F}_\infty$ with the zero-mean Gaussian measure μ_∞ whose covariance operator $\mathcal{C}_\infty$ is defined by

$$\mathcal{C}_\infty(f) \;=\; \sum_{\mathrm{u}\in\mathcal{U}} \alpha_{\mathrm{u}}^2\, C_{\mathrm{u}}(f_{\mathrm{u}}) \quad \text{for all} \quad f \;=\; \sum_{\mathrm{u}\in\mathcal{U}} f_{\mathrm{u}} \;\in\; \mathcal{F}_\infty.$$

Note that μ_∞ is well defined if and only if it has a finite trace. As shown in the Appendix of [17][1],

$$\mathrm{trace}(\mathcal{C}_\infty) \;=\; \sum_{\mathrm{u}\in\mathcal{U}} \alpha_{\mathrm{u}}^2\, \gamma_{\mathrm{u}}^2\, \mathrm{trace}(C_{\mathrm{u}}).$$

Since $\mathrm{trace}(C_{\mathrm{u}}) = (\mathrm{trace}(C_1))^{|\mathrm{u}|}$, we assume that

$$\sum_{\mathrm{u}\in\mathcal{U}} \alpha_{\mathrm{u}}^2\, \gamma_{\mathrm{u}}^2\, (\mathrm{trace}(C_1))^{|\mathrm{u}|} \;<\; \infty, \tag{2.4}$$

which is a necessary and sufficient condition for μ_∞ to be well defined.

If $\mathcal{F}_\infty$ is a reproducing kernel Hilbert space (see Remark 1) then the covariance kernel of μ_∞, as shown in the Appendix of [17], exists and is given by

$$\mathcal{C}_\infty^{\mathrm{ker}}(\boldsymbol{x},\boldsymbol{y}) \;=\; \sum_{\mathrm{u}\in\mathcal{U}} \alpha_{\mathrm{u}}^2\, \gamma_{\mathrm{u}}^2 \prod_{j\in\mathrm{u}} C_1^{\mathrm{ker}}(x_j, y_j).$$

2.3 Linear Problem

Let $\mathcal{G}$ be a normed space whose norm is denoted by $\|\cdot\|_{\mathcal{G}}$. Let $\mathcal{S}_\infty$ be a linear operator

$$\mathcal{S}_\infty \;:\; \mathcal{F}_\infty \;\to\; \mathcal{G},$$

and let

$$S_{\mathrm{u}} \;:=\; \mathcal{S}_\infty|_{F_{\mathrm{u}}} \;:\; F_{\mathrm{u}} \;\to\; \mathcal{G}$$

be the restrictions of $\mathcal{S}_\infty$ to the spaces F_{u}. Let $\|S_{\mathrm{u}}\|$ denote the corresponding operator norm,

$$\|S_{\mathrm{u}}\| \;=\; \sup_{f_{\mathrm{u}}\in F_{\mathrm{u}}} \frac{\|S_{\mathrm{u}}(f_{\mathrm{u}})\|_{\mathcal{G}}}{\|f_{\mathrm{u}}\|_{F_{\mathrm{u}}}}.$$

[1] In [17], δ_{u} is used instead of α_{u}, and α_{u} there is what $\alpha_{\mathrm{u}}\,\gamma_{\mathrm{u}}$ is in the current paper.

We assume that

$$\left[\sum_{\mathfrak{u}\in\mathcal{U}} \left(\gamma_{\mathfrak{u}} \left\|S_{\mathfrak{u}}\right\|\right)^2\right]^{1/2} < \infty. \tag{2.5}$$

This is because (2.5) implies continuity of $\mathcal{S}_\infty$. Indeed, we have the following proposition whose proof can be found in, e.g., [17][1].

Proposition 2 *The operator norm of $\mathcal{S}_\infty$ is bounded by*

$$\left\|\mathcal{S}_\infty\right\| \;\le\; \left[\sum_{\mathfrak{u}\in\mathcal{U}} \left(\gamma_{\mathfrak{u}} \left\|S_{\mathfrak{u}}\right\|\right)^2\right]^{1/2}$$

with the equality if $\mathcal{S}_\infty$ is a functional.

If the weights have product form and $\mathcal{S}_\infty$ satisfies the following condition:

$$\text{there exists constant } T_1 \text{ such that} \quad \left\|S_{\mathfrak{u}}\right\| \;\le\; T_1^{|\mathfrak{u}|} \quad \text{for all } \mathfrak{u}\in\mathcal{U}, \tag{2.6}$$

then

$$\left[\sum_{\mathfrak{u}\in\mathcal{U}} \left(\gamma_{\mathfrak{u}} \left\|S_{\mathfrak{u}}\right\|\right)^2\right]^{1/2} \;\le\; \left[\prod_{j=1}^{\infty}(1+(\gamma_j\, T_1)^2)\right]^{1/2}.$$

We will illustrate some of the results using the following function approximation problem.

Example 3 (L_q **Approximation**) Let $D=[0,1]$ and $F_{\mathfrak{u}}$ consist of

$$f_{\mathfrak{u}}(\boldsymbol{x}_{\mathfrak{u}}) \;=\; \int_{D^{|\mathfrak{u}|}} \prod_{j\in\mathfrak{u}} (x_j - t_j)_+^0 \, h(\boldsymbol{t}_{\mathfrak{u}})\mathrm{d}\boldsymbol{t}_{\mathfrak{u}} \quad \text{for} \quad h_{\mathfrak{u}} \in L_2(D^{|\mathfrak{u}|})$$

with the norm equal to the L_2 norm of $h_{\mathfrak{u}}$, i.e., the L_2 norm of the mixed partial first order derivatives of $f_{\mathfrak{u}}$. The reproducing kernel of $F_{\mathfrak{u}}$ is then

$$K_{\mathfrak{u}}(\boldsymbol{x},\boldsymbol{y}) \;=\; \prod_{j\in\mathfrak{u}} \min(x_j, y_j).$$

For S_1 consider the embedding from F_1 into $L_q([0,1])$ for given $q\in[1,\infty]$. Then

$$\left\|S_1\right\| \;\le\; T_1 \;=\; \left(\frac{2}{2+q}\right)^{1/q},$$

which reduces to 1 for $q=\infty$. Hence

$$\left\|S_{\mathfrak{u}}\right\| \;\le\; \left(\frac{2}{2+q}\right)^{|\mathfrak{u}|/q}.$$

2.4 Specific Weights

My results primarily assume *product weights* of the form

$$\gamma_{\mathfrak{u}} = \prod_{j \in \mathfrak{u}} \gamma_j \quad \text{and} \quad \alpha_{\mathfrak{u}} = \prod_{j \in \mathfrak{u}} \alpha_j$$

introduced in [14]. Here, γ_j and α_j are positive numbers and without loss of generality $\gamma_j \geq \gamma_{j+1}$ for all $j \in \mathbb{N}$. It also becomes convenient later to write

$$\overline{\gamma}_{\mathfrak{u}} = \gamma_{\mathfrak{u}} \|S_u\|, \qquad \text{and} \qquad \overline{\alpha}_{\mathfrak{u}} = \alpha_{\mathfrak{u}} \gamma_{\mathfrak{u}} \sqrt{\text{trace}(C_1)^{|\mathfrak{u}|}}.$$

For example, in my numerical results, I use $\gamma_j = j^{-a}$, $\alpha_j = j^{-b}$ for $a \in \{1, 2\}$, $b \in \{0, 1, 2\}$, but my methods work for other product weights.

Chapter 3 ε-Truncation Dimension

The method of ε-truncation seeks to reduce the infinitely many variables to a finite (and small) number, denoted by $\dim^{\text{trnc}}(\varepsilon, \mathcal{S})$. ε-Truncation in the worst case has been introduced in [7]. We extend it in [3] to average and probabilistic settings with respect to the probability measure μ_∞. Following are my methods for finding the ε-truncation dimension in these average and probabilistic cases, and the results for different parameters.

3.1 Methods

The fact that we are working with anchored spaces gives us the very important property from [12]: For any f, any $\mathfrak{u}$, and any $\boldsymbol{x}$,

$$f_{\mathfrak{u}}(\boldsymbol{x}) \;=\; \sum_{\mathfrak{w}\subseteq\mathfrak{u}} (-1)^{|\mathfrak{u}|-|\mathfrak{w}|} f([\boldsymbol{x};\mathfrak{w}]).$$

Because the variables in $\mathcal{F}_\infty$ have been weighted without loss of generality in increasing order, the first variables will give the largest contribution to our approximation of $\mathcal{S}_\infty$. Thus, we consider a non-negative integer k and the corresponding function f_k:

$$f_k(\boldsymbol{x}) \;:=\; f(x_1, \ldots, x_k, 0, 0 \ldots) \;=\; f([\boldsymbol{x}; \{1, \ldots, k\}]). \tag{3.1}$$

Both f_k and $\mathcal{S}_\infty(f_k)$ are well defined since

$$f_k \;=\; \sum_{\mathfrak{w}\subseteq\{1,\ldots,k\}} f_{\mathfrak{w}} \in \mathcal{F}_\infty.$$

From [3], we have the following definition and theorem.

Definition 4 *Let $\varepsilon > 0$ and $\delta \in (0,1)$. The average ε-truncation dimension is defined by*

$$\dim^{\text{avg-t}}(\varepsilon; \mathcal{S}_\infty, \mu_\infty) \;:=\; \inf\left(k \;:\; \int_{\mathcal{F}_\infty} \|\mathcal{S}_\infty(f - f_k)\|_{\mathcal{G}}^2 \, \mu_\infty(\mathrm{d}f) \;\le\; \varepsilon^2\right). \tag{3.2}$$

The probabilistic (ε, δ)-truncation dimension is defined by

$$\dim^{\text{prb-t}}(\varepsilon, \delta; \mathcal{S}_\infty, \mu_\infty) \;:=\; \inf\left(k \;:\; \mu_\infty\left(\{f \in \mathcal{F}_\infty \;:\; \|\mathcal{S}_\infty(f - f_k)\|_{\mathcal{G}} \le \varepsilon\}\right) \;\ge\; 1 - \delta\right). \tag{3.3}$$

In both cases we use the convention that $\inf \emptyset = \infty$.

To shorten the notation, we will write sometimes $\dim^{\text{avg-trnc}}(\varepsilon)$ and $\dim^{\text{prob-trnc}}(\varepsilon, \delta)$.

Theorem 5 *We have*

$$\dim^{\text{avg-t}}(\varepsilon) \;\le\; \inf\left(k \;:\; B(k) \;\le\; \varepsilon^2\right)$$

and

$$\dim^{\text{prb-t}}(\varepsilon,\delta) \;\le\; \inf\left(k \;:\; B(k) \;\le\; \frac{\varepsilon^2}{2\,\ln(5/\delta)}\right),$$

where

$$B(k) \;:=\; \left[\sum_{\mathfrak{u}\not\subseteq\{1,\ldots,k\}} (\gamma_{\mathfrak{u}}\,\|S_{\mathfrak{u}}\|)^2\right]\left[\sum_{\mathfrak{u}\not\subseteq\{1,\ldots,k\}} (\alpha_{\mathfrak{u}}\,\gamma_{\mathfrak{u}})^2\,(\;\mathrm{trace}(C_1))^{|\mathfrak{u}|}\right] \tag{3.4}$$

for $k\in\{0\}\cup\mathbb{N}$. *Additionally, if*

$$\alpha_{\mathfrak{u}} \;=\; \prod_{j\in\mathfrak{u}}\alpha_j, \quad \gamma_{\mathfrak{u}} \;=\; \prod_{j\in\mathfrak{u}}\gamma_j, \quad \textit{and} \quad \|S_{\mathfrak{u}}\| \;\le\; T_1^{|\mathfrak{u}|}, \tag{3.5}$$

then

$$B(k) \;\le\; \mathrm{ER}(k)\;\;\mathrm{trace}(\mathcal{C}_\infty)\prod_{j=1}^{\infty}(1+(\gamma_j\,T_1)^2)$$

where

$$\mathrm{ER}(k) \;:=\; \left[1-\exp\left(-\,T_1^2\sum_{j=k+1}^{\infty}\gamma_j^2\right)\right]\left[1-\exp\left(-\,\mathrm{trace}(C_1)\sum_{j=k+1}^{\infty}(\alpha_j\,\gamma_j)^2\right)\right].$$

I constructed a program to find the bound given by Theorem 5 for different parameters. This program assumes the restrictions given by (3.5), i.e. that the weights are of Product Form. This program was written in python and uses the Decimal library to control precision of mathematical operations.

In order to find this bound, we need to be able to compute $B(k)$. We can begin to find the value of $B(k)$ given in (3.4) by noting

$$\sum_{\mathfrak{u}\not\subseteq\{1,\ldots,k\}} x_{\mathfrak{u}} \quad=\quad \sum_{\mathfrak{u}\in\mathcal{U}} x_{\mathfrak{u}} \;-\; \sum_{\mathfrak{u}\subseteq\{1,\ldots,k\}} x_{\mathfrak{u}}.$$

The goal is, then, to approximate

$$\sum_{\mathfrak{u}\in\mathcal{U}}\gamma_{\mathfrak{u}}^2\|S_{\mathfrak{u}}\|^2 \qquad\text{and}\qquad \sum_{\mathfrak{u}\in\mathcal{U}}\alpha_{\mathfrak{u}}^2\gamma_{\mathfrak{u}}^2\mathrm{trace}(C_1)^{|\mathfrak{u}|} \tag{3.6}$$

from above with relative error significantly smaller than ε^2, and find an efficient means of calculating

$$\sum_{\mathfrak{u}\subseteq\{1,\ldots,k\}}\gamma_{\mathfrak{u}}^2\|S_{\mathfrak{u}}\|^2 \qquad\text{and}\qquad \sum_{\mathfrak{u}\subseteq\{1,\ldots,k\}}\alpha_{\mathfrak{u}}^2\gamma_{\mathfrak{u}}^2\mathrm{trace}(C_1)^{|\mathfrak{u}|}.$$

The latter is achieved using the following equality, which holds only because we assume product weights:

$$\sum_{\mathfrak{u}\subseteq\{1,\ldots,k\}} c^{|\mathfrak{u}|}\gamma_{\mathfrak{u}} = \prod_{j=1}^{k}(1+c\;\gamma_j).$$

This gives us the easier to compute

$$\sum_{\mathfrak{u}\subseteq\{1,\ldots,k\}} \gamma_{\mathfrak{u}}^2 \|S_{\mathfrak{u}}\|^2 \le \sum_{\mathfrak{u}\subseteq\{1,\ldots,k\}} \gamma_{\mathfrak{u}}^2 T_1^{2|\mathfrak{u}|} = \prod_{j=1}^{k} \left(1 + T_1^2 \frac{1}{j^{2a}}\right),$$

and

$$\sum_{\mathfrak{u}\subseteq\{1,\ldots,k\}} \alpha_{\mathfrak{u}}^2 \gamma_{\mathfrak{u}}^2 \mathrm{trace}(C_1)^{|\mathfrak{u}|} = \prod_{j=1}^{k} \left(1 + \mathrm{trace}(C_1) \frac{1}{j^{2(a+b)}}\right).$$

I use a similar approach to that of [5] to obtain approximations of (3.6):

Definition 6 *For a positive number s, let*

$$\begin{aligned}
L &:= \sum_{\mathfrak{u}\in\mathcal{U}} \left(\gamma_{\mathfrak{u}}^2 \|S_{\mathfrak{u}}\|^2\right), \\
L_s &:= \exp\left(\frac{T_1^2}{(2a-1)(s+\frac{1}{2})^{(2a-1)}}\right) \prod_{j=1}^{s} \left(1 + T_1^2 \frac{1}{j^{2a}}\right), \\
R &:= \sum_{\mathfrak{u}\in\mathcal{U}} \left(\alpha_{\mathfrak{u}}^2 \gamma_{\mathfrak{u}}^2 \mathrm{trace}(C_1)^{|\mathfrak{u}|}\right), \\
R_s &:= \exp\left(\frac{\mathrm{trace}(C_1)}{(2(a+b)-1)(s+\frac{1}{2})^{(2(a+b)-1)}}\right) \prod_{j=1}^{s} \left(1 + \mathrm{trace}(C_1) \frac{1}{j^{2(a+b)}}\right)
\end{aligned}$$

Proposition 7

$$L \le L_s \qquad \textit{and} \qquad R \le R_s$$

Proof. We assume product weights, with

$$\gamma_j = \frac{1}{j^a} \qquad \text{and} \qquad \alpha_j = \frac{1}{j^b}$$

for $a > 1/2$ and $a + b > 1/2$. Then

$$\begin{aligned}
L &= \sum_{\mathrm{u}\in\mathcal{U}} \left(\gamma_{\mathrm{u}}^2 \|S_{\mathrm{u}}\|^2\right) \\
&= \sum_{\mathrm{u}\in\mathcal{U}} \left(\|S_{\mathrm{u}}\|^2 \prod_{j\in\mathrm{u}} \frac{1}{j^{2a}}\right) \\
&\leq \sum_{\mathrm{u}\in\mathcal{U}} \left(T_1^{2|\mathrm{u}|} \prod_{j\in\mathrm{u}} \frac{1}{j^{2a}}\right) \\
&= \prod_{j=1}^{\infty} \left(1 + T_1^2 \frac{1}{j^{2a}}\right) \\
&= \exp\left(\ln\left(\prod_{j=s+1}^{\infty} \left(1 + T_1^2 \frac{1}{j^{2a}}\right)\right)\right) \prod_{j=1}^{s} \left(1 + T_1^2 \frac{1}{j^{2a}}\right) \\
&\leq \exp\left(T_1^2 \sum_{j=s+1}^{\infty} \frac{1}{j^{2a}}\right) \prod_{j=1}^{s} \left(1 + T_1^2 \frac{1}{j^{2a}}\right) \\
&\leq \exp\left(T_1^2 \int_{s+1/2}^{\infty} \frac{1}{x^{2a}} dx\right) \prod_{j=1}^{s} \left(1 + T_1^2 \frac{1}{j^{2a}}\right) \\
&= \exp\left(\frac{T_1^2}{(2a-1)(s+\frac{1}{2})^{(2a-1)}}\right) \prod_{j=1}^{s} \left(1 + T_1^2 \frac{1}{j^{2a}}\right) \;=\; L_s
\end{aligned}$$

The proof of $R \leq R_s$ is similar. □

In [3] we also prove that the relative error between L and L_s is proportional to $1/s^{4a-2}$ with asymptotic constant $T_1^2/(2^{2a-1}) \prod_{j=1}^{\infty} (1 + T_1^2/j^{2a})$. Similarly, The relative error between R and R_s is proportional to $1/s^{4(a+b)+2}$ with asymptotic constant equal to $\mathrm{trace}(C_1)/(2^{2(a+b)+1}) \prod_{j=1}^{\infty} (1 + \mathrm{trace}(C_1)/j^{2(a+b)})$

For even modest s, these L_s and R_s approximations provide an approximation of L and R with error much less than ε. In my programs, I used $s = 1000000$, as this value calculated the approximation quickly while maintaining accuracy for the smallest value of ε at 10^{-5}. I experimented with different values of s, but for $a = 2$ and most values of q, increasing s from 1000000 causes a change in L_s on the order of 10^{-30}, and our threshold from Theorem 5 is much larger than that (order of 10^{-12} at the smallest).

A naïve approach would now begin at $k = 0$ and increment k by one until the threshold defined in Theorem 5 is reached. I improve on this by beginning at $k = 2$ and doubling k until the threshold is exceeded, then using binary search to find the exact value. In addition, in my program I have parameterized the starting value of $k = 2$ to allow for quick re-tests. This was very helpful when calculating probabilistic (ε, δ)-truncation dimension for different values of δ.

The time complexity of computing the products

$$\prod_{j=1}^{k}\left(1+T_1^2\frac{1}{j^{2a}}\right), \qquad \text{and} \qquad \prod_{j=1}^{k}\left(1+\operatorname{trace}(C_1)\frac{1}{j^{2(a+b)}}\right)$$

is given by $\mathcal{O}(\mathrm{k})$. Let m be $2^{\lceil \log_2(\dim)\rceil}$. In our method, m is the largest dimension we evaluate the weight of. Thus, the time complexity of finding these weights is given by $\mathcal{O}(m)$. We evaluate the weights of $\lceil \log_2(\dim)\rceil$ weights on our way up to m, then conduct our binary search. The run time of the binary search in the worst case is given by $\mathcal{O}(m/2)$. The total time complexity of our method, then, is $\mathcal{O}(\mathrm{m}\log(\mathrm{m}))$.

3.2 Results

Using the methods outlined above, I was able to calculate the ε-truncation dimension under several different parameters:

$$\operatorname{trace}(C_1) = 2, \quad \varepsilon = 10^{-i} \text{ for } i \in \{2,\ldots,5\}, \quad \gamma_j = j^{-2} \quad \text{and} \quad \alpha_j = j^{-b} \text{ for } b \in \{0,1,2\}.$$

To obtain a bound on $\|S_1\|$, I also considered the approximation problem from Example 3 for

$$q \in \{1,2,\infty\} \quad \text{with} \quad \|S_1\| \le T_1 = \left(\frac{2}{2+q}\right)^{1/q}.$$

Of course, for $q=\infty$ we have $T_1 = 1$. I use $s = 1000000$ for my approximation of L_s and R_s.

$\varepsilon =$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$b=0$	4	9	19	42
$b=1$	3	5	8	15
$b=2$	2	3	5	8

for $q=1$,

$\varepsilon =$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$b=0$	4	9	20	43
$b=1$	3	5	9	15
$b=2$	2	3	5	9

for $q=2$,

and

$\varepsilon =$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$b=0$	5	11	24	51
$b=1$	3	6	10	18
$b=2$	2	4	6	10

for $q=\infty$.

I collected similar results for probabilistic (δ,ε)-truncation dimension. Specifically, I collected results with the same parameters above, and $\delta \in \{10^{-1}, 10^{-2}, 10^{-3}\}$. I report results for $\delta = 10^{-3}$ here:

$\varepsilon =$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$b=0$	7	14	32	67
$b=1$	4	7	12	22
$b=2$	3	4	7	11

for $q=1$,

$\varepsilon =$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$b = 0$	7	15	32	69
$b = 1$	4	7	12	22
$b = 2$	3	4	7	11

for $q = 2$,

and

$\varepsilon =$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$b = 0$	8	17	38	81
$b = 1$	4	8	14	25
$b = 2$	3	5	8	13

for $q = \infty$.

Using the same parameters, I also collected results for the execution time of my algorithm. All results were collected on the same machine, and each case was executed immediately after the last. Times are measured in seconds. The time to approximate L_s and R_s is not included in these totals. I report first the execution times for the average ε-truncation dimension.

$\varepsilon =$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$b = 0$	0.0012520	0.0011422	0.0010427	0.0016267
$b = 1$	0.0002747	0.0004818	0.0004640	0.0007422
$b = 2$	0.0000684	0.0002716	0.0004693	0.0004658

for $q = 1$,

$\varepsilon =$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$b = 0$	0.0006516	0.0007147	0.0010271	0.0016027
$b = 1$	0.0002702	0.0005822	0.0007173	0.0007102
$b = 2$	0.0000636	0.0002689	0.0004716	0.0011338

for $q = 2$,

and

$\varepsilon =$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$b = 0$	0.0005720	0.0011173	0.0010507	0.0017738
$b = 1$	0.0002671	0.0004551	0.0006951	0.0010400
$b = 2$	0.0000711	0.0002796	0.0004684	0.0006951

for $q = \infty$.

And here are the execution times for the probabilistic (δ, ε)-truncation dimension, with $\delta = 10^{-3}$:

$\varepsilon =$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$b = 0$	0.0005996	0.0015333	0.0012889	0.0027591
$b = 1$	0.0003751	0.0008378	0.0008298	0.0012067
$b = 2$	0.0003742	0.0003618	0.0014644	0.0008280

for $q = 1$,

$\varepsilon =$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$b = 0$	0.0006231	0.0008498	0.0020004	0.0027951
$b = 1$	0.0007947	0.0005982	0.0008329	0.0012640
$b = 2$	0.0003756	0.0004956	0.0006316	0.0008542

for $q = 2$,

and

$\varepsilon =$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$b = 0$	0.0007827	0.0013360	0.0017280	0.0045151
$b = 1$	0.0003720	0.0005818	0.0008516	0.0020204
$b = 2$	0.0004156	0.0011027	0.0009658	0.0010502

for $q = \infty$.

The execution times are mostly negligible, as each takes just a fraction of a second, but these give us insight into how each parameter effects execution time. It makes sense that the execution time is highly correlated with the dimension.

As my program is written in Python, it is unlikely any caching sped up the process, but it would be possible to achieve a significant increase in time by caching the weights for calculated values. When increasing k, one could start at an already-calculated value instead of 1 for the calculation of

$$\prod_{j=1}^{k}\left(1+T_1^2\frac{1}{j^{2a}}\right), \qquad \text{and} \qquad \prod_{j=1}^{k}\left(1+\operatorname{trace}(C_1)\frac{1}{j^{2(a+b)}}\right).$$

This was not necessary for my purposes, though.

Chapter 4 ε-Superposition Dimension

The concept of ε-superposition is subtly different from ε-truncation. Where ε-truncation reduces our infinitely many variables to a small finite number, ε-superposition instead considers the weights of the actual groups being constructed.

For example, with an ε-truncation dimension of 3, we only consider the groups

$$\mathfrak{u} \in \mathcal{V}, \qquad \mathcal{V} = \big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\big\}$$

However, ε-superposition dimension does not limit itself to a specific set of variables. Our method is to continuously add the set with the highest weight to our 'active set' $\mathcal{V}$, regardless of the variables therein. As a result, one possible example of a set with an ε-superposition dimension of 2 would be:

$$\mathfrak{u} \in \mathcal{V}, \qquad \mathcal{V} = \big\{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{1,2\}, \{1,3\}\big\}$$

Of course, many other possible examples of sets with ε-superposition dimension of 2 exist. In general, ε-superposition results in smaller sets and lower dimension than ε-truncation, as it can always choose the group with the largest weights and does not force itself to take possibly low-value groups early, such as $\{1,2,3\}$ in these examples.

The concept of ε-superposition dimension in the worst case setting was used implicitly by [11] when introducing what are now referred to as *Multivariate Decomposition Methods* (in that paper, *Changing Dimension Algorithms*). A very efficient method to calculate the dimension for product weights was proposed in [5]. We implement a similar method to extend this concept in average and probabilistic settings.

4.1 Definitions

The following is repeated from [3] for context. We first define

$$\|\mathcal{V}\|_\infty := \max_{\mathfrak{u}\in\mathcal{V}} |\mathfrak{u}| \quad \text{for any } \mathcal{V} \subset \mathcal{U}.$$

Definition 8 *Let $\varepsilon > 0$ and $\delta \in (0,1)$. A subset $\mathcal{V} \subseteq \mathcal{U}$ is said to be ε-active if*

$$\int_{\mathcal{F}_\infty} \left\| \mathcal{S}_\infty\Big(\sum_{\mathfrak{u}\in\mathcal{U}\setminus\mathcal{V}} f_\mathfrak{u} \Big) \right\|_{\mathcal{G}}^2 \mu_\infty(\mathrm{d}f) \le \varepsilon^2.$$

It is said to be (ε,δ)-active if

$$\mu_\infty\left(\left\{ f \in \mathcal{F}_\infty : \left\| \mathcal{S}_\infty\Big(\sum_{\mathfrak{u}\in\mathcal{U}\setminus\mathcal{V}} f_\mathfrak{u} \Big) \right\|_{\mathcal{G}} \le \varepsilon \right\} \right) \ge 1-\delta.$$

The average and probabilistic ε-superposition dimensions are defined respectively by

$$\dim^{\text{avg-s}}(\varepsilon; \mathcal{S}_\infty, \mu_\infty) := \inf\{\|\mathcal{V}\|_\infty : \mathcal{V} \text{ is } \varepsilon\text{-active}\}, \tag{4.1}$$

$$\dim^{\text{prb-s}}(\varepsilon,\delta; \mathcal{S}_\infty, \mu_\infty) := \inf\{\|\mathcal{V}\|_\infty : \mathcal{V} \text{ is } (\varepsilon,\delta)\text{-active}\}. \tag{4.2}$$

By $\mathcal{V}_\varepsilon$ or $\mathcal{V}_{(\varepsilon,\delta)}$ we denote subsets that are respectively ε-active or (ε,δ)-active whose $\|\cdot\|_\infty$ are equal to the corresponding average or probabilistic superpositions dimension.

Often we will simply write $\dim^{\text{avg-s}}(\varepsilon)$ and $\dim^{\text{prb-s}}(\varepsilon,\delta)$.

Corollary 9 *It is easy to see that*

$$\dim^{\text{avg-s}}(\varepsilon) \le \dim^{\text{avg-t}}(\varepsilon) \quad \textit{and} \quad \dim^{\text{prb-s}}(\varepsilon,\delta) \le \dim^{\text{prb-t}}(\varepsilon,\delta)$$

As we shall see, under the assumptions already introduced, there exist $\mathcal{V}_\varepsilon$ and $\mathcal{V}_{(\varepsilon,\delta)}$ subsets that are not only finite but have surprisingly small cardinalities. This property makes *Multivariate Decomposition Methods* (see [11]) so effective since it is enough to approximate only those $S_{\mathfrak{u}}(f_{\mathfrak{u}})$ whose $\mathfrak{u}$ belongs to $\mathcal{V}_e$ (or $\mathcal{V}_{(\delta,\varepsilon)}$).

Proposition 10 *Any subset $\mathcal{V} \subseteq \mathcal{U}$ satisfying*

$$\left[\sum_{\mathfrak{u}\in\mathcal{U}\setminus\mathcal{V}} \gamma_{\mathfrak{u}}^2 \,\|S_{\mathfrak{u}}\|^2\right]\left[\sum_{\mathfrak{u}\in\mathcal{U}\setminus\mathcal{V}} (\alpha_{\mathfrak{u}}\,\gamma_{\mathfrak{u}})^2 \,\mathrm{trace}(C_{\mathfrak{u}})\right] \le \varepsilon^2 \tag{4.3}$$

is ε-active. Any subset $\mathcal{V} \subseteq \mathcal{U}$ satisfying

$$\left[\sum_{\mathfrak{u}\in\mathcal{U}\setminus\mathcal{V}} \gamma_{\mathfrak{u}}^2 \,\|S_{\mathfrak{u}}\|^2\right]\left[\sum_{\mathfrak{u}\in\mathcal{U}\setminus\mathcal{V}} (\alpha_{\mathfrak{u}}\,\gamma_{\mathfrak{u}})^2 \,\mathrm{trace}(C_{\mathfrak{u}})\right] \le \frac{\varepsilon^2}{2\,\ln(5/\delta)} \tag{4.4}$$

is (ε,δ)-active.

4.2 Methods

It is these bounds (4.3) and (4.4) that I am most concerned with in my work.

We already have a good approximation of

$$\sum_{\mathfrak{u}\in\mathcal{U}} \gamma_{\mathfrak{u}}^2 \|S_{\mathfrak{u}}\|^2 = \sum_{\mathfrak{u}\in\mathcal{U}} \overline{\gamma}_{\mathfrak{u}}^2 \qquad \text{and} \qquad \sum_{\mathfrak{u}\in\mathcal{U}} \alpha_{\mathfrak{u}}^2\gamma_{\mathfrak{u}}^2 \mathrm{trace}(C_1)^{|\mathfrak{u}|} = \sum_{\mathfrak{u}\in\mathcal{U}} \overline{\alpha}_{\mathfrak{u}}^2$$

from Definition 6. I follow the efficient method from [5] and extend it to the average and probabilistic settings by using these new bounds defined in (4.3) and (4.4). Roughly, we begin with our approximations L_s and R_s from Definition 6 and continuously add sets $\mathfrak{u}$ to $\mathcal{V}$, subtracting from L_s and R_s as we go until we have achieved the bounds defined in (4.3) and (4.4).

Selecting which $\mathfrak{u}$ to add to $\mathcal{V}$ poses a problem, though. We would like to add those $\mathfrak{u} \in \mathcal{U}$ with the largest impact on $L_s \cdot R_s$ first, if possible. The impact adding a given set $\mathfrak{w}$ to $\mathcal{V}$ will have on our bounds (4.3) and (4.4) is given by

$$\begin{aligned}
&\left[\sum_{\mathfrak{u}\in\mathcal{U}\setminus\mathcal{V}} \overline{\gamma}_{\mathfrak{u}}^2\right]\left[\sum_{\mathfrak{u}\in\mathcal{U}\setminus\mathcal{V}} \overline{\alpha}_{\mathfrak{u}}^2\right] - \left[\sum_{\mathfrak{u}\in\mathcal{U}\setminus(\mathcal{V}\cup\mathfrak{w})} \overline{\gamma}_{\mathfrak{u}}^2\right]\left[\sum_{\mathfrak{u}\in\mathcal{U}\setminus(\mathcal{V}\cup\mathfrak{w})} \overline{\alpha}_{\mathfrak{u}}^2\right] \\
&= \overline{\gamma}_{\mathfrak{w}}^2\left[\sum_{\mathfrak{u}\in\mathcal{U}\setminus\mathcal{V}} \overline{\alpha}_{\mathfrak{u}}^2\right] + \overline{\alpha}_{\mathfrak{w}}^2\left[\sum_{\mathfrak{u}\in\mathcal{U}\setminus\mathcal{V}} \overline{\gamma}_{\mathfrak{u}}^2\right] - \overline{\gamma}_{\mathfrak{w}}^2\overline{\alpha}_{\mathfrak{w}}^2.
\end{aligned} \tag{4.5}$$

Comparing this impact value between different weights quickly becomes exhausting, as each time we update $\mathcal{V}$ we have to recalculate the impact for each other set. Thus, instead of adding sets one-by-one, I take inspiration from [5] and add many sets at once, as long as we remain above the threshold. To do this, I divide $\mathbb{R}_+$ into intervals I_i such that the numbers in I_j are greater than those in I_{j+1}. After trying a few different intervals, I settled on using

$$I_1 = [2^{-1}, \infty), \quad \text{and} \quad I_j = [2^{-j}, 2^{-j+1}) \quad \text{for } j = 2, 3, \ldots,$$

though I believe a better, less uniform partition is possible. Instead of adding each $\mathfrak{u}$ one at a time, I gather all the $\mathfrak{u}$ whose impact value falls into a single interval and add them all at once, as long as it does not bring us below the thresholds defined by (4.3) and (4.4). This significantly speeds up the process of constructing $\mathcal{V}$.

One important piece of this program that I've omitted so far is generating new sets to add. The method I use is identical to that of [5]. We first add the empty set to $\mathcal{V}$, as it always has the largest weight. We consider the non-empty sets in order of increasing cardinality. Hence, I start with singleton sets $\mathfrak{u} = \{1\}, \{2\}, \{3\}, \ldots$, as long as their 'impact' defined above is within the current interval. Once I reach a set not in this interval, I store it in a list of sets to consider for the next interval and move on to sets with cardinality 2. I repeat the same process, beginning with $\mathfrak{u} = \{1, 2\}, \{1, 3\}, \{1, 4\}, \ldots$ and then $\mathfrak{u} = \{2, 3\}, \{2, 4\}, \{2, 5\}, \ldots$ and so on. The same process is continued for sets with increasing cardinality until we have scoped out all groups that belong to the current interval. We can then add all these sets to $\mathcal{V}$ at once if this does not bring us below our threshold, or we can add them one at a time within this interval to get right over the threshold and not add extra sets. If we do not reach our threshold, we simply continue with the next interval.

4.3 Results

I have computed the average and probabilistic ε-superposition dimensions for the same parameters as for ε-truncation dimension, i.e.,

$$\operatorname{trace}(C_1) = 2, \quad \varepsilon = 10^{-i} \text{ for } i \in \{2, \ldots, 5\}, \quad \gamma_j = j^{-2} \quad \text{and} \quad \alpha_j = j^{-b} \text{ for } b \in \{0, 1, 2\}$$

with $T_1 = (2/(2+q))^{1/q}$ for $q = 1, 2, \infty$. In the following tables I list the dimension followed by the cardinality of the constructed active set. First, the average ε-superposition dimension:

$\varepsilon =$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$b = 0$	2 and 9	3 and 28	4 and 84	4 and 296
$b = 1$	2 and 5	2 and 10	3 and 25	3 and 57
$b = 2$	2 and 4	2 and 13	2 and 13	3 and 23

for $q = 1$,

$\varepsilon =$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$b = 0$	2 and 9	3 and 29	4 and 94	4 and 314
$b = 1$	2 and 5	2 and 11	3 and 25	3 and 58
$b = 2$	2 and 4	2 and 6	3 and 13	3 and 23

for $q = 2$,

and

$\varepsilon =$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$b = 0$	3 and 12	3 and 40	4 and 134	4 and 424
$b = 1$	2 and 6	3 and 13	3 and 31	4 and 74
$b = 2$	2 and 4	2 and 7	3 and 14	3 and 28

for $q = \infty$,

As before, I also took results for probabilistic ε-superposition dimension with $\delta \in \{10^{-1}, 10^{-2}, 10^{-3}\}$. Below are results for $\delta = 10^{-3}$:

$\varepsilon =$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$b = 0$	3 and 18	3 and 57	4 and 182	4 and 601
$b = 1$	2 and 8	3 and 18	3 and 40	4 and 93
$b = 2$	2 and 5	2 and 9	3 and 18	3 and 35

for $q = 1$,

$\varepsilon =$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$b = 0$	3 and 19	3 and 60	4 and 192	4 and 625
$b = 1$	2 and 8	3 and 18	3 and 42	4 and 99
$b = 2$	2 and 5	2 and 9	3 and 19	3 and 35

for $q = 2$,

and

$\varepsilon =$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$b = 0$	3 and 25	4 and 84	4 and 277	5 and 886
$b = 1$	2 and 9	3 and 22	3 and 53	4 and 126
$b = 2$	2 and 5	3 and 11	3 and 21	3 and 242

for $q = \infty$,

It is easy to see from these results why approximation methods based on ε-superposition are so efficient. The most difficult case listed above is the probabilistic superposition dimension for $\delta = 10^{-3}$, $\varepsilon = 10^{-5}$, $\alpha_j \equiv 1$. Even then, we need only approximate 886 functions, each with at most 5 variables, in order to approximate the whole function f with error, say $2 \cdot 10^{-5}$.

Using the same parameters, following are the execution times, measured in seconds, of my algorithm for each case. These were all computed on the same machine immediately after one another. The times for calculating L_s and R_s are not included in these totals. For the average setting:

$\varepsilon =$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$b = 0$	0.0006467	0.0016587	0.0045111	0.0109676
$b = 1$	0.0003640	0.0006511	0.0013604	0.0028449
$b = 2$	0.0002298	0.0003662	0.0007716	0.0011942

for $q = 1$,

$\varepsilon =$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$b = 0$	0.0005227	0.0015507	0.0045907	0.0112924
$b = 1$	0.0003569	0.0006249	0.0013369	0.0027947
$b = 2$	0.0002027	0.0003862	0.0007013	0.0012462

for $q = 2$,

and

$\varepsilon =$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$b = 0$	0.0006498	0.0022596	0.0054698	0.0154444
$b = 1$	0.0003404	0.0007524	0.0014804	0.0034618
$b = 2$	0.0001782	0.0004298	0.0006653	0.0014609

for $q = \infty$,

For $\delta = 10^{-3}$, these are the execution times for the probabilistic ε-superposition dimension:

$\varepsilon =$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$b = 0$	0.0012307	0.0032693	0.0071449	0.0211893
$b = 1$	0.0006716	0.0012044	0.0021142	0.0049244
$b = 2$	0.0005453	0.0006142	0.0011702	0.0018916

for $q = 1$,

$\varepsilon =$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$b = 0$	0.0011556	0.0029947	0.0088484	0.0218658
$b = 1$	0.0017182	0.0013698	0.0019520	0.0053609
$b = 2$	0.0005738	0.0006058	0.0011280	0.0020147

for $q = 2$,

and

$\varepsilon =$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$b = 0$	0.0015018	0.0037382	0.0096893	0.0309280
$b = 1$	0.0012707	0.0013204	0.0033027	0.0051756
$b = 2$	0.0003951	0.0007644	0.0012911	0.0021436

for $q = \infty$,

Bibliography

[1] R. E. Caflish, M. Morokoff, and A. B. Owen, Valuation of mortgage backed securities using Brownian bridges to reduce effective dimension, *J. Comp. Finance* **1** (2009), 27-46.

[2] H. Daumé III, From zero to reproducing kernel hilbert spaces in twelve pages or less. *University of Maryland*, (2004).

[3] J. Dingess and G. W. Wasilkowski, ε-Superposition and Truncation Dimensions in Average and Probabilistic Setting for ∞-Variate Linear Problems, in progress

[4] A. Gilbert, F. Y. Kuo, D. Nuyens, and G. W. Wasilkowski, Efficient implementations of the Multivariate Decomposition Method for approximating infinite-variate integrals, *SIAM Journal on Scientific Computation*, to appear, 2019.

[5] A. Gilbert and G. W. Wasilkowski, Small superposition dimension and active set construction for multivariate integration under modest error demand, *J. Complexity* **42** (2017), 94–109.

[6] A. Hinrichs, P. Kritzer, F. Pillichshammer, and G. W. Wasilkowski, Truncation dimension for linear problems on multivariate function spaces, *Numerical Algorithms* to appear.

[7] P. Kritzer, F. Pillichshammer, and G. W. Wasilkowski, Very low truncation dimension for high dimensional integration under modest error demand, *J. of Complexity* **35** (2016), 63–85.

[8] P. Kritzer, F. Pillichshammer, and G. W. Wasilkowski, Truncation in average and worst case settings for special classes of ∞-variate functions, *Mathematics and Computers in Simulation*, to appear.

[9] F. Y. Kuo and D. Nuyens, Application of quasi-Monte Carlo methods to elliptic PDE's with random diffusion coefficients - a survey of analysis and implementation, *Found. Comput. Math.* **16** (2016) 1631–1696.

[10] F. Y. Kuo, Ch. Schwab, and I. H. Sloan, Quasi-Monte Carlo finite element methods for a class of elliptic partial differential equations with random coefficients, *SIAM J. Numer. Anal.* **50** (2012), 3351–3374.

[11] F. Y. Kuo, I. H. Sloan, G. W. Wasilkowski, and H. Woźniakowski, Liberating the dimension, *J. Complexity* **26** (2010), 422–454.

[12] F. Y. Kuo, I. H. Sloan, G. W. Wasilkowski, and H. Woźniakowski, On decomposition of multivariate functions, *Mathematics of Computation* **79** (2010), 953–966.

[13] L. Plaskota and G. W. Wasilkowski, Tractability of infinite-dimensional integration in the worst case and randomized settings, *J. Complexity* **27** (2011), 505–518.

[14] I. H. Sloan and H. Woźniakowski, When are quasi-Monte Carlo methods efficient for high-dimensional integrals? *J. Complexity* **14** (1998), 1–33.

[15] J. F. Traub, G. W. Wasilkowski, and H. Woźniakowski, Information-Based Complexity, *Academic Press,*, NY 1988.

[16] X. Wang and K.-T. Fang, Effective dimension and quasi-Monte Carlo integration, *J. of Complexity* **19** (2003), 101-124.

[17] G. W. Wasilkowski, Average case tractability of approximating ∞-variate functions, *Mathematics of Computation*, **83** (2014), 1319–1336.

Vita

Jonathan Dingess

Education

- Bachelor of Science in Computer Science with a minor in Mathematics at University of Kentucky, August 2015 - January 2019
- Master of Science in Computer Science at University of Kentucky, August 2015 - January 2019. Thesis Title: "ε-Superposition and Truncation Dimensions in Average and Probabilistic Settings for ∞-variate Linear Problems"

Recent Work Experience

- Intern at **Noblis** (Summer of 2017, Summer of 2018)
- Software Developer at **Noblis** (June 2019 - Present)

Honors and Achievements

- National Merit Finalist
- University of Kentucky Patterson Scholarship Winner
- University of Kentucky Dean's List
- Graduation Magna Cum Laude at the University of Kentucky in 2019
- Red Cross Life Guard Certification
- PADI Open Water Diver Certification
- 2-dan American Go Association Ranking